Waves and Jets in a Viscous Liquid.

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1. In an able article which appeared in the ninth volume of this Journal, Prof. Greenhill discussed at considerable length the principal cases of wave-motion in a *frictionless* liquid which have hitherto been solved. In the present paper, I propose to consider certain problems of a similar character when the viscosity of the liquid is taken into account.

The motion of deep-sea waves is worked out in Chapter XXIII of my Treatise on Hydrodynamics, from which it appears that when the viscosity is small, as is the case with water, the velocity of propagation is the same as that of a frictionless liquid, and that the only effect of viscosity is to introduce a modulus of delay whose value is equal to $\lambda^2/8\pi^2\nu$, where λ is the wave-length and ν the kinematic coefficient of viscosity. But when the viscosity is large the motion is of a totally different character, the time factor appearing in the form of an exponential quantity ε^{kt} where k is *negative.* The result which is given in §521 of my book is wrong; for it can be shown, and will hereafter appear from the solution of another problem, that the correct value* of k is $-g/2m\nu$, whence the form of the free surface at time t after the commencement of the motion is

$$\zeta = (a \cos mx + b \sin mx)\,\varepsilon^{-\frac{gt}{2m\nu}},$$

where $2\pi/m$ is the wave-length, and a and b are constants depending upon the initial motion. The difference between the two kinds of motion may be illustrated by allowing a jet of air to play for a short time upon the surface of water and upon treacle, and watching the effect produced.

The motion of waves in a viscous liquid of finite depth is worked out in §523 of my Hydrodynamics. The solution is more complicated, but leads to similar results.

* See also Tait, Proc. Roy. Soc. Edinburgh, 1890, p. 110.

The motion of waves at the surface of separation of two frictionless liquids which are moving with independent velocities was investigated by Greenhill* in 1878, and his solution has been discussed by Lord Rayleigh† with reference to the question of stability, but the corresponding solution for *two* viscous liquids would entail difficulties of a rather formidable character, owing to the fact that it would be necessary to retain all quadratic terms upon which the undisturbed motion depends. If, however, one of the liquids is frictionless whilst the other is viscous, the solution can be arrived at without difficulty.

It is always a great advantage when the conditions of a mathematical problem can be expressed in terms of a single function, which satisfies a certain partial differential equation together with given boundary conditions. In most problems of interest relating to wave-motion in frictionless liquids this can be affected by means of the velocity potential; but when the liquid is viscous, a velocity potential cannot as a rule exist, owing to the fact that the motion almost always involves molecular rotation. Of course any solution giving a possible irrotational motion must necessarily satisfy the equations of motion of a viscous liquid; but the solution will not satisfy the boundary conditions except in the single case in which the liquid moves like a rigid body *having a motion of translation alone.* Moreover, viscous motion involves a conversion of energy into heat. If, however, the motion of a viscous liquid is in two-dimensions, or is symmetrical with respect to an axis, the former can always be expressed by means of the current function ψ; and this function will be used throughout the present paper.

Waves at the Surface of Separation of Two Liquids.

2. Let the origin be some point in the undisturbed surface of separation; let the axis of x be measured in the direction of propagation of the waves, and the axis of z vertically upwards. Let the upper liquid be viscous and moving with constant velocity U_1, whilst the lower is frictionless and moving with constant velocity U_2. Let μ be the viscosity, and let the suffixes 1 and 2 refer to the upper and lower liquids respectively.

Earnshaw's current function ψ satisfies the equations

$$\left(\nu\nabla^2 - \frac{d}{dt}\right)\nabla^2\psi = \left(u\frac{d}{dx} + w\frac{d}{dz}\right)\nabla^2\psi, \tag{1}$$

* Cambridge Mathematical Tripos Examination.

† Proc. London Math. Soc., Vols. X, XI and XIX.

where

$$u = d\psi/dz, \quad w = -d\psi/dx, \tag{2}$$

and ν is the kinematic coefficient of viscosity which is equal to μ/ρ.

Since the liquid possesses an independent velocity U_1, the solution of (1) must be of the form

$$\psi = \psi' + U_1 z,$$

where ψ' depends on the disturbed motion. Substituting in (1) and neglecting all quadratic terms which depend *solely* on the disturbed motion, we obtain

$$\left(\nu\nabla^2 - \frac{d}{dt} - U_1\frac{d}{dx}\right)\nabla^2\psi' = 0. \tag{3}$$

To obtain a solution suitable for representing wave-motion we assume that x and t enter in the form of the factor $\varepsilon^{\iota mx + kt}$, where $2\pi/m$ is the wave-length and k is a quantity to be determined. Substituting in (3) and putting

$$\alpha^2 = m^2 + (k + \iota m U_1)/\nu, \tag{4}$$

we obtain

$$\left(\frac{d^2}{dz^2} - m^2\right)\left(\frac{d^2}{dz^2} - \alpha^2\right)\psi' = 0, \tag{5}$$

where ψ' now denotes a function of z alone.

The solution of (5) is

$$\psi' = \chi_1 + \chi_2,$$

where χ_1, χ_2 satisfy the equations

$$\left.\begin{aligned} \frac{d^2\chi_1}{dz^2} - m^2\chi_1 &= 0, \\ \frac{d^2\chi_2}{dz^2} - \alpha^2\chi_2 &= 0. \end{aligned}\right\} \tag{6}$$

3. We shall consider the case in which the depths of the two liquids are so large in comparison with the wave-length that the former may be treated as infinite; under these circumstances the proper solutions of (6) will be

$$\chi_1 = A\varepsilon^{-mz}, \quad \chi_2 = C\varepsilon^{-\alpha z},$$

so that

$$\psi_1 = (A\varepsilon^{-mz} + C\varepsilon^{-\alpha z})\,\varepsilon^{\iota mx + kt} + U_1 z. \tag{7}$$

The hydrodynamical equations for the pressure are

$$\frac{du}{dt} + U_1 \frac{du}{dx} = -\frac{1}{\rho_1}\frac{dp_1}{dx} + \nu\nabla^2 u,$$
$$\frac{dw}{dt} + U_1 \frac{dw}{dx} = -\frac{1}{\rho_1}\frac{dp_1}{dz} - g + \nu\nabla^2 w;$$

from these combined with (2), (3) and (6) we readily obtain

$$\left.\begin{aligned}(k + \iota m U_1)\frac{d\chi_1}{dz} &= -\frac{1}{\rho_1}\frac{dp_1}{dx}, \\ \iota m (k + \iota m U_1)\chi_1 &= \frac{1}{\rho_1}\frac{dp_1}{dz} + g,\end{aligned}\right\} \tag{8}$$

whence

$$p_1 = \text{const.} - g\rho_1 z - \iota\rho_1 (k + \iota m U_1) A\varepsilon^{-mz + \iota mx + kt}. \tag{9}$$

Since the lower liquid is supposed to be frictionless, the function ψ_2 satisfies the equation

$$\nabla^2\psi_2 = 0,$$

whence

$$\psi_2 = B\varepsilon^{mz + \iota mx + kt} + U_2 z \tag{10}$$

and

$$p_2 = \text{const.} - g\rho_2 z + \iota\rho_2 (k + \iota m U_2) B\varepsilon^{mz + \iota mx + kt}. \tag{11}$$

Having obtained the equations of motion and the pressure, we must now consider the boundary conditions which hold good at the surface of separation.

Let the equation of this surface be

$$\zeta - a\varepsilon^{\iota mx + kt} = F(\zeta, x, t) = 0, \tag{12}$$

then the kinematical condition which holds good at every bounding surface is

$$\frac{dF}{dt} + \frac{d\psi}{dz}\frac{dF}{dx} - \frac{d\psi}{dx}\frac{dF}{d\zeta} = 0. \tag{13}$$

Applying this to the upper liquid we get

$$a = -\frac{\iota m (A + C)}{k + \iota m U_1}. \tag{14}$$

Applying (13) to the lower liquid we get

$$a = -\frac{\iota m B}{k + \iota m U_2}. \tag{15}$$

The dynamical conditions which hold good at the boundary are continuity of stress; and since a frictionless liquid is incapable of supporting any tangential

stress or shear, it follows that the tangential stress parallel to the plane $z = 0$ must vanish at the surface of separation. This condition gives

$$\frac{du}{dz} + \frac{dw}{dx} = 0,$$

or

$$\frac{d^2\psi_1}{dz^2} - \frac{d^2\psi_1}{dx^2} = 0, \tag{16}$$

when $z = 0$.

To obtain the second condition, consider a film of the viscous liquid bounded on the lower side by the surface of separation and on the upper side by a surface indefinitely near the former. The lower surface of the film will be subjected to the pressure p_2 of the frictionless liquid, and the upper to a *traction* R due to the action of the viscous liquid in contact with it; hence if T be the surface tension between the two liquids, we get

$$p_2 + R = \frac{T}{\rho} = -T\frac{d^2\zeta}{dx^2}, \tag{17}$$

since the curvature ρ^{-1} of the film is supposed to be very small.

Substituting from (7) in (16) we obtain

$$2Am^2 + C(\alpha^2 + m^2) = 0. \tag{18}$$

The value of R is

$$\begin{aligned} R &= -p_1 + 2\mu\frac{dw}{dz} = -p_1 - 2\rho_1\nu\frac{d^2\psi_1}{dxdz} \\ &= g\rho_1 a + \iota\rho_1(k + \iota m U_1)A + 2\iota\rho_1 m\nu(Am + C\alpha) \end{aligned}$$

by (9); whence by (11) and (12), (17) becomes

$$\begin{aligned} g(\rho_1 - \rho_2)a + \iota\rho_1(k + \iota m U_1 + 2m^2\nu)A \\ + 2\iota\rho m\alpha\nu C + \iota\rho_2(k + \iota m U_2)B - Tm^2 a = 0. \end{aligned} \tag{19}$$

The elimination of a, A, C and B between (14), (15), (18) and (19) leads to an equation for determining k.

From (4), (14) and (18) we obtain

$$\iota A = -a(k + \iota m U_1 + 2m^2\nu)/m, \quad \iota C = 2am\nu;$$

whence the result of the elimination is

$$mg(\rho_1 - \rho_2) - \rho_1(k + \iota m U_1 + 2m^2\nu)^2 + 4\rho_1 m^3\alpha\nu^2 - \rho_2(k + \iota m U_2)^2 - Tm^3 = 0. \tag{20}$$

Equation (20) combined with (4) gives a biquadratic equation for determining k, and we shall first perform the elimination.

Let
$$x = k + \iota m U_1 \tag{21}$$
and
$$I = (\rho_1 + \rho_2)\, x^2 + 2x\{2\rho_1 m^2 \nu + \iota \rho_2 m (U_2 - U_1)\} \\ + mg(\rho_2 - \rho_1) + Tm^3 - \rho_2 m^2 (U_2 - U_1)^2, \tag{22}$$
then (20) becomes
$$I = 4\rho_1 m^3 \nu^2 (\alpha - m);$$
also (4) may be written
$$x = \nu(\alpha^2 - m^2),$$
whence
$$I^2 + 8I\rho_1 m^4 \nu^2 - 16\rho_1^2 m^6 \nu^3 x = 0. \tag{23}$$
This is the biquadratic equation which we have to discuss.

Oil Floating on Water.

4. The viscosity of water is equal to the tangential stress of .014 dynes per square centimetre; and since a dyne is approximately equal to the attraction of the earth at its surface upon a cubic millimetre of water,* the viscosity of water is a very small quantity. On the other hand the viscosity of olive oil is about 3.25 dynes per square centimetre, and is therefore about 232 times as great as that of water. If, therefore, a thick layer of oil is floating on water, the former may be treated as a highly viscous liquid and the latter as a frictionless one, and consequently the proper solution for our purpose will be one in which the viscosity is supposed to be a large quantity.

Assuming for trial that x varies as ν^{-1}, it follows that the first term and also the imaginary term in I are negligible; accordingly (23) reduces to
$$I = 2\rho_1 m^2 \nu x,$$
which gives, since $\nu\rho_1 = \mu$,
$$k + \iota m U_1 = -\frac{g(\rho_2 - \rho_1) + Tm^2 - m\rho_2 (U_2 - U_1)^2}{2m\mu}. \tag{24}$$

(i). Let $U_1 = U_2 = 0$, so that the liquids are initially at rest; then, since $\rho_2 > \rho_1$, k is a negative quantity and wave-motion in the proper sense of the word does not exist. The amplitude of the initial disturbance gradually diminishes with the time and the disturbed motion ultimately dies away.

* More accurately the 981st part of a cubic centimetre.

When the wave-length is large, m or $2\pi/\lambda$ is small, in which case the term depending on gravity is the most important; on the other hand when the wave-length is small, the term depending on the surface tension is the most important. It therefore follows that the damping effect of gravity is most important in the case of long waves, and that of surface tension in the case of short ones.

We have supposed that the upper liquid is oil and the lower one water, so that $\rho_2 > \rho_1$; if, however, $\rho_1 > \rho_2$ so that the upper liquid is the heaviest, the value of k will be positive unless

$$Tm^2 > g(\rho_1 - \rho_2),$$

and consequently unless this inequality is satisfied the equilibrium will be unstable. If, however, the difference of densities is very small and the waves are sufficiently short, it will be possible for this inequality to be satisfied, and we shall have the curious case of a heavier liquid floating on a lighter one in stable equilibrium. The stability will, however, be confined to disturbances of very short wave-length.

When the wave-length is such that the numerator of (24) vanishes, a critical case arises. This case is best discussed by reverting to the original differential equation for k. It will be found that one root is zero and that another is real and negative, so that there must be one stable mode of motion, but this case is not of any particular interest.

(ii). When $\rho_2 > \rho_1$ and U_1, U_2 are not zero, it follows that the real part of k will be positive provided the term $(U_2 - U_1)^2$ is large enough. From this it follows that a high relative velocity, which includes the case of motion in opposite directions, tends to produce instability. This has been verified experimentally by Prof. Osborne Reynolds.* The exact nature of the instability will depend upon whether the two liquids are capable of permanently mixing. If they are, the subsequent motion will at first be of a highly turbulent character, which will continue until the liquids have become thoroughly mixed together; but if they are incapable of remaining permanently mixed, a layer of turbulent liquid will exist in the neighborhood of the surface of separation, resembling a mixture of oil and water which has been thoroughly well shaken together. If the liquids be left to themselves, the damping effect of viscosity will gradually cause the turbulent motion to subside and the ultimate state will be one in which all relative motion between the parts of the viscous liquid will have disappeared.

* Phil. Trans., 1883.

It must, however, be recollected that as the lower liquid is supposed to be frictionless, it is not necessary that the tangential velocities on either side of the surface of separation should ultimately become equal.

The critical case in which the numerator of (24) vanishes would require separate treatment.

5. We shall now consider the case in which the viscosity is very small, and we shall suppose that the *lower* liquid is viscous. This renders it necessary to reverse the sign of g. Since ν is very small, the last two terms of (23) may be neglected, and the equation for k is

$$(\rho_1+\rho_2)\,x^2+2x\,\{2\rho_1 m^2\nu+\iota\rho_2 m\,(U_2-U_1)\}\\ +mg\,(\rho_1-\rho_2)+Tm^3-\rho_2 m^2\,(U_2-U_1)^2=0. \quad (25)$$

If the sum of the last three terms is positive, it follows that the roots of this equation must be of the form

$$\alpha+\iota\beta,\quad \gamma\,(\alpha-\iota\beta),$$

where γ is a positive quantity; also

$$4\rho_1 m^2\nu=-\alpha\,(1+\gamma),$$

which shows that α must be negative; hence the real part of k will be negative, and the motion will be stable, provided

$$g\,(\rho_1-\rho_2)+Tm^2>\rho_2 m\,(U_2-U_1)^2.$$

If this inequality is not satisfied γ will be negative, and therefore one root of k must be positive, so that the motion will be unstable.

When both liquids are at rest and $\rho_1>\rho_2$, the real part of k is equal to $-2\rho_1 m^2\nu/(\rho_1+\rho_2)$, which shows that the amplitude diminishes with the time, and the wave-motion gradually dies away.

Motion of a Sheet of Liquid.

6. In §418 of my book, a short investigation is given respecting the motion of a thin sheet of frictionless liquid. We shall now discuss the same problem when the liquid is viscous, and surface tension is taken into account; but in order to avoid unnecessarily complicating the analysis, the effect of gravity will be neglected. The conditions of the problem may be approximately realized in practice by supposing liquid to escape from a long and narrow horizontal slit in

the side of a cistern, and confining our attention to that portion of the sheet which is nearly horizontal.

Let U be the undisturbed velocity of the sheet, $2l$ its thickness. We shall also suppose that the disturbed motion is such, that the sinuosities are symmetrical with respect to the middle surface of the sheet.

The value of ψ will be

$$\psi = \chi_1 + \chi_2 + Uz,$$

where χ_1, χ_2 are determined by (6) of §2. Since we suppose that the motion is symmetrical with respect to the plane $z=0$, $d\psi/dz$ must be an even function of z, and therefore

$$\psi = (A \sinh mz + C \sinh \alpha z)\, \varepsilon^{\iota mx + kt} + Uz, \tag{1}$$

whence from (8) of §3 we obtain

$$p = \iota\rho\,(k + \iota m U)\, A \cosh mz\, \varepsilon^{\iota mx + kt} \tag{2}$$

Let the equation of the free surface of the upper side of the sheet be

$$\zeta - l - a\varepsilon^{\iota mx + kt} = F = 0,$$

then applying (12) of §3, we get

$$a = -\frac{\iota m\,(A \sinh ml + C \sinh \alpha l)}{k + \iota m U}. \tag{3}$$

The condition of zero tangential stress gives

$$2Am^2 \sinh ml + C(\alpha^2 + m^2) \sinh \alpha l = 0. \tag{4}$$

Again, omitting the exponential factor,

$$\begin{aligned} R &= -p - 2\nu\rho \frac{d^2\psi}{dx dz} \\ &= \iota\rho\,(k + \iota m U)\, A \cosh ml - 2\iota\rho m\nu\,(Am \cosh ml + C\alpha \cosh \alpha l). \end{aligned}$$

The surface tension condition is

$$R = -\frac{T}{\rho'} = T\frac{d^2\zeta}{dx^2},$$

which becomes

$$\iota\rho\,(k + \iota m U + 2m^2\nu)\, A \cosh ml + 2\iota\rho\nu m\alpha C \cosh \alpha l = Tm^2 a. \tag{5}$$

On applying the same conditions to the lower side of the jet, where $z = -l - a\varepsilon^{\iota mx + kt}$, it will be found that they lead to exactly the same equations; hence on eliminating a, A and C we obtain

$$(k + \iota m U + 2m^2\nu)^2 \coth ml - 4m^3\nu^2\alpha \coth \alpha l + \frac{Tm^3}{\rho} = 0. \tag{6}$$

This equation combined with (4) of §2 determines the values of k.

When ν is very small, $\alpha^2 = (k + \iota m U)/\nu$ approximately; α is therefore a large quantity, and $\coth \alpha l = 1$. Also $\alpha\nu^2$ is very small, so that the second term of (6) may be neglected. Under these circumstances we obtain

$$k = -2m^2\nu - \iota m U \pm \iota\sqrt{\frac{Tm^3}{\rho}\tanh ml}. \tag{7}$$

From this equation we see that the real part of k, which is proportional to the viscosity, is negative, and therefore the motion is stable, from which it follows that the tendency of viscosity is in the direction of stability.

We also see that the effect of surface tension is to introduce an imaginary term into the value of k, which consequently does not affect the stability. In the case of a *circular* jet, it is known that surface tension produces instability when the disturbance consists of short waves. It is remarkable that surface tension should produce such dissimilar effects; and it is important to notice that the effects, produced by certain causes upon a liquid whose motion is in two dimensions, are often totally different from those produced by the same causes when the motion is in three dimensions. The former kind of motion is generally the simplest from a mathematical point of view, whilst the latter kind more often occurs in practical and experimental applications. Great caution must therefore be observed in drawing inferences from theoretical results derived from the solutions of problems in two-dimensional motion and in applying them to the explanation of experimental phenomena in which the motion is in three dimensions.

When ν is absolutely zero, equation (7) gives the value of k for a frictionless liquid. This result can be verified by an independent investigation.

Equation (6) may be written in the form

$$(k + \iota m U)^2 \coth ml + 4m^2\nu (k + \iota m U) \coth ml$$
$$+ 4m^3\nu^2 (m \coth ml - \alpha \coth \alpha l) + \frac{Tm^3}{\rho} = 0.$$

When ν is very large, α is nearly equal to m, so that we may write

$$\alpha = m + (k + \iota m U)/2m\nu$$

approximately; whence, by Taylor's theorem,

$$l\alpha \coth \alpha l = lm \coth ml + \frac{(k + \iota m U)\, l}{2m\nu} (\coth ml - ml \operatorname{cosech}^2 ml),$$

and the equation for k becomes

$$(k + \iota m U)^2 \coth ml + 2m^3 l\nu (k + \iota m U) \operatorname{cosech}^2 ml + Tm^3/\rho = 0,$$

which gives
$$k + \iota m U = -\frac{T \sinh^2 ml}{2\mu l}$$
approximately. From this result we see that when the viscosity is large, its effect, combined with that of the surface tension, is to cause the disturbed motion to die away.

Motion of a Cylindrical Jet.

7. The motion of a cylindrical jet of *frictionless* liquid was discussed by Lord Rayleigh about ten years ago;* and in two recent papers† he has returned to the subject and considered the case of a *highly viscous* liquid. I now propose to give an investigation of a far more general character, in which the velocity, viscosity, surface tension and also the influence of the surrounding air are taken into account. I shall in addition suppose the jet to be electrified.

When a fine jet is allowed to escape from a nozzle so as to describe an approximately parabolic path, it is observed that at a certain point the jet becomes disintegrated into drops. If, however, a metal ball, which is so feebly electrified that it scarcely causes the leaves of a gold-leaf electrometer to diverge to any perceptible extent, is brought into the neighborhood of the jet, the disintegration ceases. It is, moreover, necessary for the electrification to be feeble, for a powerful charge causes the disintegration to proceed more rapidly. I believe that Lord Rayleigh considers that this phenomenon cannot be completely explained by ordinary mechanical methods, but is mainly due to the fact that the electrification causes the drops to coalesce instead of rebounding upon collision. At the same time very little complication is introduced into the work by supposing the jet to be electrified, and the results are instructive, as they give information as to the effect which a charge is likely to produce. We shall find that the charge tends to produce stability or instability according as the ratio of the circumference of the jet to the wave-length of the disturbance is less or greater than .6.

8. We shall suppose throughout that the jet consists of a straight cylindrical column of liquid, and that the disturbance is symmetrical with respect to the axis. Taking the axis of the jet as the direction of the axis of z, and measuring

* Proc. Lond. Math. Soc., Vols. X, XI and XIX.

† Phil. Mag., Aug., 1892.

r perpendicularly to it, the motion can be determined by Stokes' current function, which, in the case of a viscous liquid, satisfies the equation

$$\left(\nu D - \frac{d}{dt}\right) D\psi = \left(u \frac{d}{dr} + w \frac{d}{dz} - \frac{2u}{r}\right) D\psi, \tag{1}$$

where

$$D = \frac{d^2}{dz^2} + \frac{d^2}{dr^2} - \frac{1}{r}\,\frac{d}{dr}$$

and

$$u = -\frac{1}{r}\,\frac{d\psi}{dz}, \quad w = \frac{1}{r}\,\frac{d\psi}{dr}. \tag{2}$$

When the motion is undisturbed, $w = W$, where W is the constant velocity of the jet; whence, from the second of (2),

$$\psi = \tfrac{1}{2} Wr^2,$$

which is readily seen to be a solution of (1).

Let ψ' be the portion of ψ which depends upon the disturbed motion, so that

$$\psi = \tfrac{1}{2} Wr^2 + \psi'.$$

Substituting in (1) and neglecting all quadratic terms which depend *solely* upon the disturbed motion, we obtain

$$\left(\nu D - \frac{d}{dt} - W\frac{d}{dz}\right) D\psi' = 0. \tag{3}$$

In order to obtain a solution which represents wave-motion, we assume that z and t enter in the form of the factor $\varepsilon^{\iota mz + kt}$, whence if

$$\alpha^2 = m^2 + (k + \iota m W)/\nu \tag{4}$$

(3) becomes

$$\left(\frac{d^2}{dr^2} - \frac{1}{r}\,\frac{d}{dr} - m^2\right)\left(\frac{d^2}{dr^2} - \frac{1}{r}\,\frac{d}{dr} - \alpha^2\right)\psi' = 0, \tag{5}$$

the solution of which is

$$\psi' = \chi_1 + \chi_2,$$

where

$$\left.\begin{aligned} \frac{d^2\chi_1}{dr^2} - \frac{1}{r}\,\frac{d\chi_1}{dr} - m^2\chi_1 &= 0, \\ \frac{d^2\chi_2}{dr^2} - \frac{1}{r}\,\frac{d\chi_2}{dr} - \alpha^2\chi_2 &= 0. \end{aligned}\right\} \tag{6}$$

In the first of (6) write $\chi_1 = r\phi$, and the equation for ϕ becomes

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\,\frac{d\phi}{dr} - \left(m^2 + \frac{1}{r^2}\right)\phi = 0.$$

The solution of this equation is

$$\phi = AI_1(mr) + BK_1(mr),$$

where I_1, K_1 are the two associated Bessel's functions of order unity, and are connected with the functions of order zero by the equations

$$I_0' = I_1, \quad K_0' = K_1, \tag{7}$$

where the accent denotes differentiation with respect to the argument mr. These functions are discussed in §§266–269 of my Hydrodynamics, and also in Chapter XIX of Lord Rayleigh's treatise on Sound.

The function $I_0(x)$ is expressible in the form of the series

$$I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \frac{x^6}{2^2.4^2.6^2} + \ldots., \tag{8}$$

whence

$$I_1(x) = \frac{x}{2} + \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} + \ldots.. \tag{9}$$

whilst the function $K_0(x)$ can be expressed by means of either of the series

$$K_0(x) = (\gamma + \log \tfrac{1}{2}x)\, I_0(x) - \frac{x^2}{2^2} S_1 - \frac{x^4}{2^2.4^2} S_2 - \frac{x^6}{2^2.4^2.6^2} S_6 - \quad \ldots., \tag{10}$$

or

$$K_0(x) = -\left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \varepsilon^{-x} \left\{1 - \frac{1^2}{1.8x} + \frac{1^2.3^2}{1.2(8x)} - \frac{1^2.3^2.5^2}{1.2.3.(8x)^3} - \right\} \ldots., \tag{11}$$

where γ is Euler's constant, which is equal to .5772 . . . ., and

$$S_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots. \frac{1}{n}.$$

Since the I functions are finite when $x = 0$, they are suitable for space inside a cylinder; on the other hand the K functions are infinite when $x = 0$ and zero when $x = \infty$, and are therefore suitable for space outside. It therefore follows that the proper value of ψ is

$$\psi = r\{AI_1(mr) + CI_1(\alpha r)\}\, \varepsilon^{\iota mz + kt} + \tfrac{1}{2} Wr^2. \tag{12}$$

In taking account of the influence of the surrounding atmosphere we shall neglect its compressibility and viscosity; whence, if ψ_1 be the current function,

$$D\psi_1 = 0$$

and

$$\psi_1 = BrK_1(mr)\, \varepsilon^{\iota mz + kt}. \tag{13}$$

The hydrodynamical equations for the pressure are

$$\frac{du}{dt} + W\frac{du}{dz} = -\frac{1}{\rho}\frac{dp}{dr} + \nu\left(\nabla^2 u - \frac{u}{r^2}\right),$$
$$\frac{dw}{dt} + W\frac{dw}{dz} = -\frac{1}{\rho}\frac{dp}{dz} + \nu\nabla^2 w.$$

The first of these equations may be written

$$\frac{\iota m}{r}(k + \iota m W)\psi' = \frac{1}{\rho}\frac{dp}{dr} + \iota m\nu\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - m^2\right)\frac{\psi'}{r}$$
$$= \frac{1}{\rho}\frac{dp}{dr} + \frac{\iota m\nu}{r}\left(\frac{d^2\psi'}{dr^2} - \frac{1}{r}\frac{d\psi'}{dr} - m^2\psi'\right),$$

whence
$$\frac{1}{\rho}\frac{dp}{dr} = \frac{\iota m\nu}{r}(\alpha^2 - m^2)\chi_1$$

and
$$p = \iota\rho(k + \iota m W) A I_0(mr) + \text{const.}, \tag{14}$$

which can also be shown to satisfy the second equation.

In the same way the pressure of the air can be shown to be

$$p_1 = \iota\rho_1 k B K_0(mr) + \text{const.} \tag{15}$$

9. We must now consider the boundary conditions.

Let the equation of the surface of separation be

$$a + c\varepsilon^{\iota mz + kt} - r = F = 0, \tag{15A}$$

then the kinematical condition at this surface is

$$\frac{dF}{dt} - \frac{1}{r}\frac{d\psi}{dz}\frac{dF}{dr} + \frac{1}{r}\frac{d\psi}{dr}\frac{dF}{dz} = 0. \tag{16}$$

Applying this to the jet we obtain

$$c = -\frac{\iota m\{AI_1(ma) + CI_1(\alpha a)\}}{k + \iota m W}. \tag{17}$$

Applying (16) to the surrounding atmosphere we get

$$c = -\frac{\iota m B K_1(ma)}{k}. \tag{18}$$

The dynamical condition of zero tangential stress gives

$$\frac{du}{dz}+\frac{dw}{dr}=0,$$

or

$$\frac{d^2\psi'}{dr^2}-\frac{1}{r}\frac{d\psi'}{dr}+m^2\psi'=0,$$

or

$$2m^2\chi_1+(\alpha^2+m^2)\chi_2=0,$$

and therefore

$$2m^2AI_1(ma)+(\alpha^2+m^2)CI_1(\alpha a)=0. \tag{19}$$

To find the remaining boundary condition, let V be the potential of the induced charge; then the resultant force which acts *outward* on an element of the jet is

$$-\frac{1}{2}\sigma\frac{dV}{dr}=\frac{1}{8\pi}\left(\frac{dV}{dr}\right)^2,$$

also the portion due to surface tension measured outwards is

$$-T\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}-\frac{1}{a}\right),$$

where ρ_1 is the radius of curvature of the meridian section of the disturbed surface, and ρ_2 the corresponding quantity for the perpendicular section. If therefore we consider the equilibrium of a film of the disturbed surface, the required condition is easily seen to be

$$P+p_1+T\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}-\frac{1}{a}\right)=\frac{1}{8\pi}\left(\frac{dV}{dr}\right)^2. \tag{20}$$

Now

$$P=-p+2\mu\frac{du}{dr}$$

$$=-p-2\iota\rho m\nu\frac{d}{dr}\left(\frac{\psi'}{r}\right).$$

Substituting the value of p from (14) and the value of ψ' from (12), and recollecting that

$$I_1'(x)=I_0''=-\frac{I_1}{x}+I_0,$$

we obtain

$$P=-\iota\rho(k+\iota mW+2m^2\nu)AI_0(ma)+2\iota\rho m\nu a^{-1}AI_1(ma)$$
$$-2\iota\rho m\nu a^{-1}\{\alpha aI_0(\alpha a)-I_1(\alpha a)\}C+\text{const.}, \tag{21}$$

the exponential term being omitted for brevity.

Again, from (15A),

$$\frac{1}{\rho_1} = -\frac{d^2r'}{dz^2} = m^2c,$$

$$\frac{1}{\rho_2} - \frac{1}{a} = \frac{1}{a+c} - \frac{1}{a} = -\frac{c}{a^2},$$

whence the surface tension term becomes

$$T(m^2a^2 - 1)c/a^2. \tag{22}$$

The value of V is

$$V = -2E\log r + \frac{FK_0(mr)}{K_0(ma)}\epsilon^{\iota mz + kt},$$

where E is the charge on the undisturbed jet per unit of length, and F is a constant to be determined from the condition that V is constant when r is given by (15A). Dropping the exponential factor, this condition gives

$$F = 2Ec/a,$$

whence at the surface

$$-\frac{dV}{dr} = \frac{2E}{a+c} - \frac{2Ec}{a^2}\frac{maK_0'(ma)}{K_0(ma)},$$

and therefore

$$\left(\frac{dV}{dr}\right)^2 = \frac{4E^2}{a^2} - \frac{4E^2c}{a^3}\left\{1 + \frac{maK_0'(ma)}{K_0(ma)}\right\}. \tag{23}$$

Substituting from (21), (15), (22) and (23) in (20), and equating to zero the coefficients of the omitted exponential term, we finally obtain

$$\begin{aligned} -\iota\rho(k + \iota mW + 2m^2\nu)AI_0(ma) + 2\iota\rho m\nu a^{-1}AI_1(ma) \\ - 2\iota\rho m\nu a^{-1}C\{\alpha aI_0(\alpha a) - I_1(\alpha a)\} + \iota\rho_1 kBK_0(ma) \\ + T(m^2a^2 - 1)c/a^2 + \frac{E^2c}{4\pi a^2}\left\{1 + \frac{maK_0'(ma)}{K_0(ma)}\right\} = 0. \end{aligned} \tag{24}$$

The elimination of the four constants A, C, B and c between (17), (18), (19) and (24) furnishes the required equation for determining k. The result of the elimination gives the following equation for k:

$$\begin{aligned} (k + \iota mW + 2m^2\nu)^2\frac{I_0(ma)}{mI_1(ma)} - \frac{2\nu}{a}(k + \iota mW + 2m^2\nu) \\ - 4m^2\nu^2\alpha\frac{I_0(\alpha a)}{I_1(\alpha a)} + \frac{4m^2\nu^2}{a} - \frac{k^2\rho_1K_0(ma)}{\rho mK_1(ma)} + \frac{T(m^2a^2-1)}{\rho a^2} \\ + \frac{E^2}{4\pi\rho a^2}\left\{1 + \frac{maK_1(ma)}{K_0(ma)}\right\} = 0, \end{aligned} \tag{25}$$

where ρ_2 denotes the density of the atmosphere.

10. We shall now discuss this equation when the viscosity is so small that powers of ν above the first may be neglected.

Let $q = k + \iota mW$, $x = ma$; then from (4) it follows that $\alpha = (q/\nu)^{\frac{1}{2}}$ approximately, provided the wave-length is not so small that m^2 is comparable with q/ν. Equation (25) may now be written

$$\frac{q^2 I_0(x)}{xI_1(x)} + \frac{2\nu}{a^2}\left\{\frac{2xI_0(x)}{I_1(x)} - 1\right\}q + \frac{4m^2\nu^2}{a^2}\left\{\frac{xI_0(x)}{I_1(x)} - \frac{a\alpha I_0(a\alpha)}{I_1(a\alpha)}\right\}$$
$$- \frac{k^2\rho_1 K_0(x)}{\rho x K_1(x)} + \frac{T(x^2-1)}{\rho a^3} + \frac{E^2}{4\pi\rho a^3}\left\{1 + \frac{xK_1(x)}{K_0(x)}\right\} = 0. \quad (26)$$

Now if α be very large,

$$I_0(\alpha) = D\varepsilon^{\alpha}\alpha^{-\frac{1}{2}}\left\{1 + \frac{1}{2.4\alpha} + \frac{1^2.3^2}{2.4.(4\alpha)^2} + \ldots\right\},$$

from which it follows that the limit of I_0/I_1 when $\alpha = \infty$ is unity. The third term of (26) accordingly does not involve any lower power of ν than $\nu^{\frac{3}{2}}$, and may therefore be neglected.

11. We shall now consider the effect of the last three terms of (26) *separately.*

(i). When T and E are zero, (26) may be written in the form

$$\frac{k^2}{\rho}\left\{\frac{\rho I_0(x)}{xI_1(x)} - \frac{\rho_1 K_0(x)}{xK_1(x)}\right\} + \frac{2\nu k}{a^2}\left\{\frac{2xI_0(x)}{I_1(x)} - 1\right\} + \frac{2\iota WkI_0(x)}{xI_1(x)} - \frac{W^2 I_0(x)}{xI_1(x)} = 0. \quad (27)$$

Now I_0 and I_1 are always positive, but the quantity K_0/K_1 is always negative, whence the coefficient of k^2 must be positive; also from the series for I_0, I_1 it follows that the second term of (27) is also positive, whilst the last term is negative; whence the roots of (27) must be of the form $p(1+\iota\beta)$, $\gamma(1-\iota\beta)$, where the two quantities p and γ are of opposite sign. Hence the real part of one of the values of k is positive, and therefore the motion is unstable.

(ii). The fifth term of (26) depends upon the surface tension. If x is greater than unity, this term is positive, and consequently the real part of k must be negative and the motion is stable. If the liquid is supposed to be frictionless, so that ν is absolutely zero, we fall back on the case discussed by Lord Rayleigh, in which it is shown that the motion of a jet will be stable or unstable according as the ratio of its circumference to the wave-length of the disturbance is greater or less than unity.

When $x < 1$, the fifth term of (26) is negative, and consequently one value

of k must be positive; but in order to examine the case of values of x which are very slightly less than unity, it would be necessary to take into account the third term of (26) which has been neglected. This would be troublesome, but there is little doubt that its effect would be found to be that motion for which x is slightly less than unity would be stable; in other words, the effect of viscosity would be to render the liquid less unstable than if it were frictionless.

(iii). We must next consider the electrical term.

The functions $K_0(x)$ and $xK_1(x)$ have been tabulated by Lord Rayleigh.* From this table it appears that $K_0(x)$ is equal to $-\infty$, $-.9244$, and $-.7774$ for $x=\infty$, .5 and .6 respectively, whilst the corresponding values of $xK_1(x)$ are 1, .8283 and .7817. From these figures we see that xK_1/K_0 is zero when $x=0$, and becomes equal to -1 for a value of x which is slightly less than .6. Accordingly for disturbances whose wave-length is large in comparison with the circumference of the jet, the charge tends to produce stability, but for disturbances of short wave-length the charge tends to produce instability.

(iv). The effect of a very small viscosity considered by itself is to produce stability, for in this case one value of q is zero, whilst the other is

$$q=-\frac{2\nu x^2}{a^2}\left\{2-\frac{I_1(x)}{xI_0(x)}\right\}.$$

The conclusions that we draw from the preceding analysis are the following. Viscosity tends to produce stability; the influence of the surrounding air tends to produce instability; the surface tension tends to produce stability or instability, according as the ratio of the circumference of the jet to the wave-length is greater or less than unity; the electrical charge tends to produce stability or instability according as the above ratio is less or greater than .6.

These results do not entirely explain why it is that a very *slight* charge produces stability, and it must be admitted that the phenomena which are actually observed depend upon some cause of which account has not been taken, but the results do appear to give a satisfactory explanation of why it is that a large charge produces instability, for the charge produces instability for wave-lengths for which $x>.6$, and when it is sufficiently great its effect will overcome the steadying effects of viscosity and also of surface tension.

The effect of a large viscosity has been discussed by Lord Rayleigh, so that it is unnecessary to say anything upon this point.

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* Phil. Mag., Aug., 1892, p. 179. His $\phi(x)$ and $\phi'(x)$ correspond to my $K_0(x)$ and $K_1(x)$.